Modul e-V

VECTOR CALCULAS

Course outcome KAS-103T (CO-V)

Apply the concept of vector for evaluating directional derivatives, tangent and normal planes, line, surface and volume integrals

Lecture 1

SCALAR POINT FUNCTION

If for each point P of a region R, there corresponds a scalar denoted by $f(P)$, then f is called a "scalar point function" for the region R.

Example . The temperature $f(P)$ at any point P of a certain body occupying a certain region R is a scalar point function.

Example . The distance of any point $P(x, y, z)$ in space from a fixed point (x_0, y_0, z_0) is a scalar function.

$$f(P) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}.$$

Scalar field

Scalar field is a region in space such that for every point P in this region, the scalar function f associates a scalar $f(P)$.

VECTOR POINT FUNCTION

If for each point P of a region R, there corresponds a vector $\vec{f}(P)$ then $\vec{f}$ is called "vector point function" for the region R.

Example. If the velocity of a particle at a point P, at any time t be $\vec{f}(P)$, then $\vec{f}$ is a vector point function for the region occupied by the particle at time t.

If the coordinates of P be (x, y, z) then

$$\vec{f}(P) = f_1(x, y, z)\, i + f_2(x, y, z)\, j + f_3(x, y, z)\, k.$$

Vector field

Vector field is a region in space such that with every point P in the region, the vector function $\vec{f}$ associates a vector $\vec{f}(P)$.

Del operator: The linear vector differential (Hamiltorian) operator "del" defined and denoted as

$$\nabla = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

This operator also read nabla. It is not a vector but combines both differential and vectorial properties analogous to those of ordinary vectors.

GRADIENT OR SLOPE OF SCALAR POINT FUNCTION

If $f(x, y, z)$ be a scalar point function and continuously differentiable then the vector

$$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$ is called the gradient of f and is written as grad f.

Thus $$\boxed{\text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = \vec{\nabla} f}$$

GEOMETRICAL MEANING OF GRADIENT, NORMAL

Consider any point P in a region throught which a scalar field $f(x, y, z) = c$ defined. Suppose that $\vec{\nabla} f \neq 0$ at P and that there is a f = const. surface S through P and a tangent plane T; for instance, if f is a temperature field, then S is an isothermal surface (level surface). If $\hat{n}$, at P, is choosen as any vector in the tangent plane T, then surely $\frac{df}{dS}$ must be zero.

Since $$\frac{df}{dS} = \vec{\nabla} f \cdot \hat{n} = 0$$

for every $\hat{n}$ at P in the tangent plane, and both $\vec{\nabla} f$ and $\hat{n}$ are non-zero, it follows that $\vec{\nabla} f$ is normal to the tangent plane T and hence to the surface S at P.

If letting $\hat{n}$ be in the tangent plane, we learn that $\vec{\nabla} f$ is normal to S, then to seek additional information about $\vec{\nabla} f$ it seems logical to let $\hat{n}$ be along the normal line at P,.

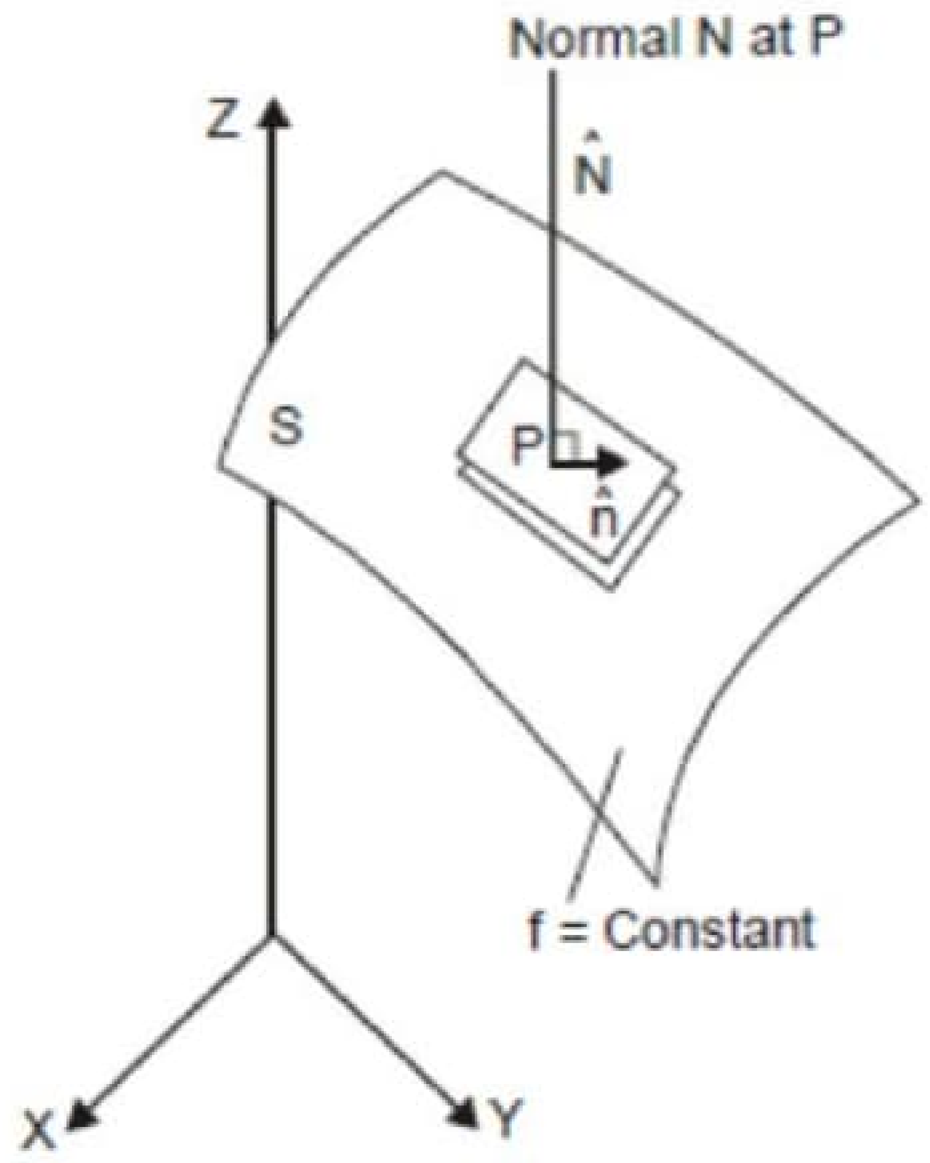


Fig. 5.2

Then $$\frac{df}{ds} = \frac{df}{dN}, \quad \hat{N} = \hat{n} \text{ then}$$

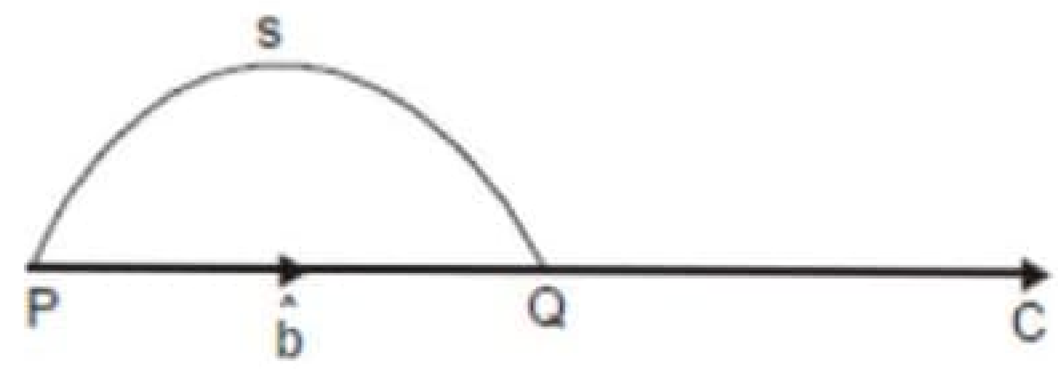
$$\frac{df}{dN} = \vec{\nabla} f \cdot \hat{N} = |\vec{\nabla} f| \cdot 1 \cos 0 = |\vec{\nabla} f| \cdot$$

So that the magnitude of $\vec{\nabla} f$ is the directional derivative of f along the normal line to S, in the direction of increasing f.

Hence, "The gradient $(\vec{\nabla} f)$ of scalar field $f(x, y, z)$ at P is vector normal to the surface f = const. and has a magnitude is equal to the directional derivative $\frac{df}{dN}$ in that direction.

DIRECTIONAL DERIVATIVE

Let $f = f(x, y, z)$ then the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the derivatives (rates of change) of f in the direction of the coordinate axes OX, OY, OZ respectively. This concept can be extended to define a derivative of f in a "given" direction $\overrightarrow{PQ}$.



Let P be a point in space and $\hat{b}$ be a unit vector from P in the given direction. Let s be the are length measured from P to another point Q along the ray C in the direction of $\hat{b}$. Now consider

$$f(s) = f(x, y, z) = f\{x(s), y(s), z(s)\}$$

Then $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \quad ...(i)$$

Here $\frac{df}{ds}$ is called directional derivative of f at P in the direction $\hat{b}$ which gives the rate of change of f in the direction of b.

Since, $$\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} = \hat{b} = \text{unit vector} \quad ...(ii)$$

Eqn. (i) can be rewritten as

$$\frac{df}{ds} = \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$\frac{df}{ds} = \left[\left(\hat{i}\frac{\partial}{\delta x} + \hat{y}\frac{\partial}{\delta y} + \hat{k}\frac{\partial}{\delta z}\right)f\right]\cdot\hat{b} = \nabla f\cdot\hat{b} \quad ...(iii)$$

Thus the directional derivative of f at P is the component (dot product) of ∇f in the direction of (with) unit vector $\hat{b}$.

Hence the directional derivative in the direction of any unit vector $\vec{a}$ is

$$\boxed{\frac{df}{ds} = \nabla f\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right)}$$

Normal derivative $\frac{df}{dn} = \nabla f\cdot\hat{n}$, where $\hat{n}$ is the unit normal to the surface f = constant.

Example If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

(*i*) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector

(*ii*) grad $r = \dfrac{\vec{r}}{r}$

(*iii*) grad $\dfrac{1}{r} = -\dfrac{\vec{r}}{r^3}$

(*iv*) grad $r^n = nr^{n-2}\,\vec{r}$, where $r = |\vec{r}|$.

Sol. (*i*) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$

then $\vec{a}\cdot\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = a_1x + a_2y + a_3z.$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1i + a_2j + a_3k = \vec{a}.$$ **Hence proved.**

(*ii*)
$$\text{grad } r = \Delta r = \Sigma\hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2}$$

$$= \Sigma\hat{i}\frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \Sigma\hat{i}\frac{x}{r} = r$$

Hence,
$$\text{grad } r = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r}.$$

(*iii*)
$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\vec{r} = \frac{-1}{r^2}\frac{\vec{r}}{r}$$

$$= -\frac{\vec{r}}{r^3}.$$ **Proved.**

(*iv*) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now,
$$\text{grad } r^n = \nabla r^n = \Sigma\hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2}$$

$$= n\,(x^2 + y^2 + z^2)^{n/2-1}\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n(x^2 + y^2 + z^2)^{(n-1)/2}\,\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= nr^{n-1}\frac{\vec{r}}{r}$$

$$= nr^{n-2}\vec{r}.$$

Example If $f = 3x^2y - y^3z^2$, find grad f at the point $(1, -2, -1)$.

Sol.

$$\text{grad } f = \vec{\nabla} f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= i\frac{\partial}{\partial x}(3x^2y - y^3z^2) + j\frac{\partial}{\partial y}(3x^2y - y^3z^2) + k\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= i(6xy) + j(3x^2 - 3y^2z^2) + k(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = i(6)(1)(-2) + j\,[(3)(1) - 3(4)(1)] + k(-2)(-8)(-1)$$

$$= -12i - 9j - 16k.$$

Lecture 2

DIVERGENCE OF A VECTOR POINT FUNCTION

If $\vec{f}$ (x, y, z) is any given continuously differentiable vector point function then the divergence of $\vec{f}$ scalar function defined as

$$\nabla \cdot \vec{f} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\vec{f} = i\cdot\frac{\partial \vec{f}}{\partial x}+j\cdot\frac{\partial \vec{f}}{\partial y}+k\cdot\frac{\partial \vec{f}}{\partial z} = div\,\vec{f}$$.

PHYSICAL INTERPRETATION OF DIVERGENCE

Let $\vec{v} = v_x i + v_y j + v_z k$ be the velocity of the fluid at $P(x, y, z)$.

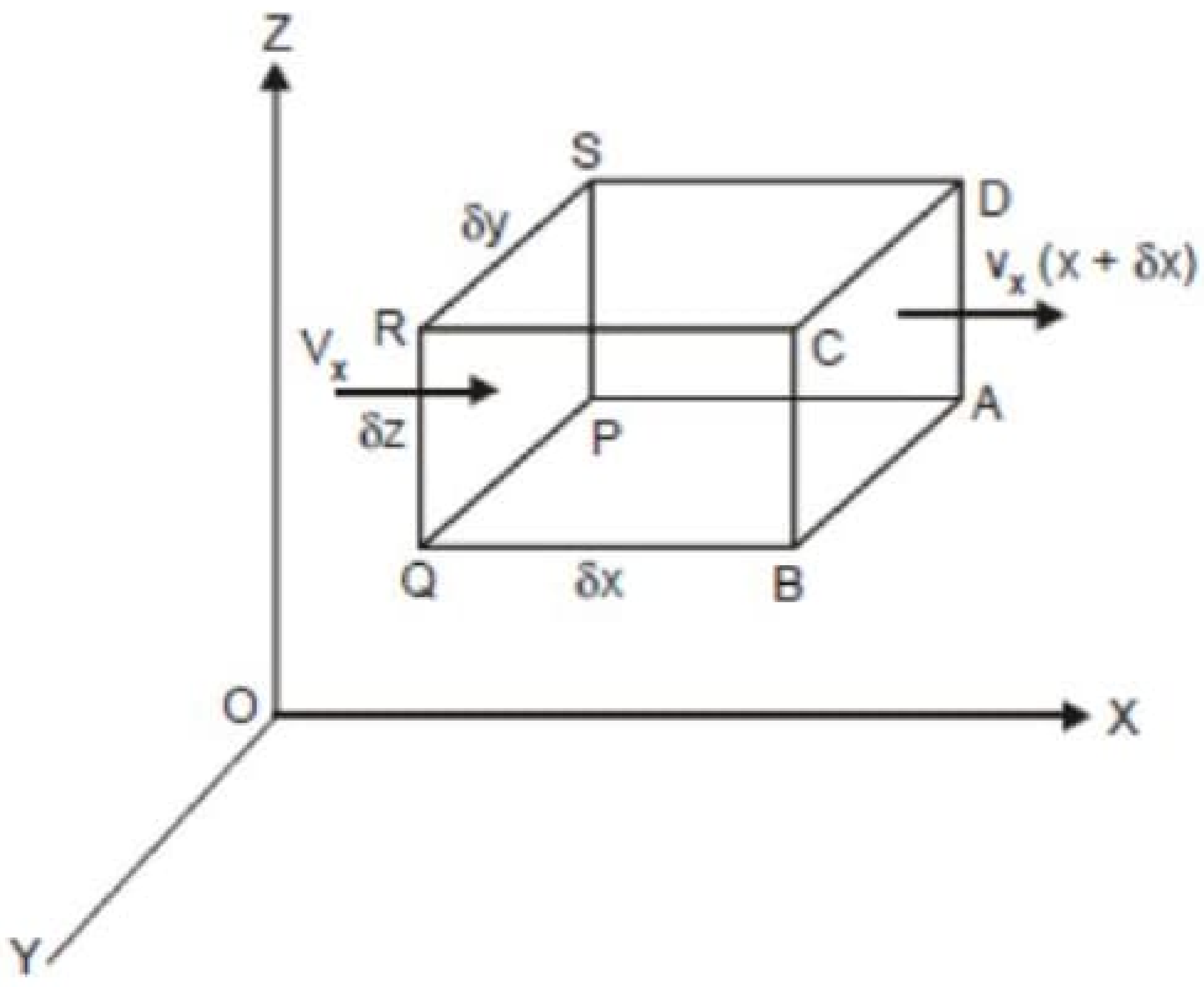
Here we consider the case of fluid flow along a rectangular parallelopiped of dimensions δx, δy, δz

Mass in $= v_x\, \delta y \delta z$ (along x-axis)

Mass out $= v_x(x + \delta x)\, \delta y \delta z$

$$= \left(v_x + \frac{\partial v_x}{\partial x}\delta x\right)\delta y \delta z$$

| By Taylor's theorem



Net amount of mass along x-axis

$$= v_x\, \delta y \delta z - \left(v_x + \frac{\partial v_x}{\partial x}\delta x\right)\delta y \delta z$$

$$= -\frac{\partial v_x}{\partial x}\,\delta x \delta y \delta z$$

| Minus sign shows decrease.

Similar net amount of mass along y-axis

$$= -\frac{\partial v_y}{\partial y}\,\delta x \delta y \delta z$$

and net amount of mass along z-axis = $-\frac{\partial v_z}{\partial z}\delta x\delta y\delta z$

$\therefore$ Total amount of fluid across parallelopiped per unit time = $-\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\delta x\delta y\delta z$

Negative sign shows decrease of amount

$\Rightarrow$ Decrease of amount of fluid per unit time = $\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\delta x\delta y\delta z$

Hence the rate of loss of fluid per unit volume

$$= \left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

$$= \nabla\cdot\vec{v} = \text{div } \vec{v}.$$

Therefore, div $\vec{v}$ represents the rate of loss of fluid per unit volume.

Solenoidal: For compressible fluid there is no gain no loss in the volume element

$\therefore$ *div* $\vec{v} = 0$

then $\vec{v}$ is called Solenoidal vector function.

CURL OF A VECTOR

If $\vec{f}$ is any given continuously differentiable vector point function then the curl of $\vec{f}$ (vector function) is defined as

$$\boxed{\text{Curl}\vec{f} = \nabla\times\vec{f} = i\times\frac{\partial\vec{f}}{\partial x}+j\times\frac{\partial\vec{f}}{\partial y}+k\times\frac{\partial\vec{f}}{\partial z}}$$

Let $\vec{f} = f_x i + f_y j + f_z k$, then

$$\nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}.$$

PHYSICAL MEANING OF CURL

Here we consider the relation $\vec{v} = \vec{w} \times \vec{r}$, $\vec{w}$ is the angular velocity $\vec{r}$ is position vector of a point on the rotating body

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v}$$

$$= \vec{\nabla} \times (\vec{w} \times \vec{r})$$

$$= \vec{\nabla} \times [(w_1 i + w_2 j + w_3 k) \times (xi + yj + zk)] \qquad \begin{bmatrix} \vec{w} = w_1 i + w_2 j + w_3 k \\ \vec{r} = xi + yj + zk \end{bmatrix}$$

$$= \vec{\nabla} \times \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times [(w_2 z - w_3 y) i - (w_1 z - w_3 x) j + (w_1 y - w_2 x) k]$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times [(w_2 z - w_3 y) i - (w_1 z - w_3 x)\, j + (w_1 y - w_2 x) k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= (w_1 + w_1)\, i - (-w_2 - w_2)\, j + (w_3 + w_3)\, k$$

$$= 2\,(w_1 i + w_2 j + w_3 k) = 2\vec{w}$$

Curl $\vec{v} = 2\vec{w}$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

Irrotational vector: If curl $\vec{f} = 0$, then the vector $\vec{f}$ is said to be irrotational. Vice-versa, if $\vec{f}$ is irrotational then, curl $\vec{f} = 0$.

Example If $\vec{f} = xy^2 i + 2x^2yz\, j - 3yz^2 k$ then find div $\vec{f}$ and curl $\vec{f}$ at the point (1, – 1, 1).

Sol. We have $\vec{f} = xy^2 i + 2x^2yz\, j - 3yz^2 k$

$$\text{div } \vec{f} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$= (-1)^2 + 2(1)^2(1) - 6(-1)(1) \quad \text{at } (1, -1, 1)$$

$$= 1 + 2 + 6 = 9.$$

Again, $\text{curl } \vec{f} = \text{curl } [xy^2 i + 2x^2yz\, j - 3yz^2 k]$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right\} + j\left\{\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right\}$$

$$+ k\left\{\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right\}$$

$$= i\,[-3z^2 - 2x^2y] + j\,[0 - 0] + k\,[4xyz - 2xy]$$

$$= (-3z^2 - 2x^2y)i + (4xyz - 2xy)\,k$$

$$= \{-3(1)^2 - 2(1)^2(-1)\}\, i + \{4(1)(-1)(1) - 2(1)(-1)\}\, k \text{ at } (1, -1, 1)$$

$$= -i - 2k.$$

Lecture 3

VECTOR IDENTITIES

Identity 1: $\text{grad } uv = u \text{ grad } v + v \text{ grad } u$

Proof:

$$\text{grad }(uv) = \nabla (uv)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(uv)$$

$$= \hat{i}\frac{\partial}{\partial x}(uv) + \hat{j}\frac{\partial}{\partial y}(uv) + \hat{k}\frac{\partial}{\partial z}(uv)$$

$$= i\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) + j\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) + k\left(u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z}\right)$$

$$= u\left(i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}\right) + v\left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)$$

or

$$\boxed{\text{grad } uv = u \text{ grad } v + v \text{ grad } u}.$$

Identity 2: $\boxed{\text{grad }(\vec{a}\cdot\vec{b}) = \vec{a} \times \text{curl } \vec{b} + \vec{b} \times \text{curl } \vec{a} + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}}$

Proof:

$$\text{grad }(\vec{a}\cdot\vec{b}) = \Sigma i\frac{\partial}{\partial x}(\vec{a}\cdot\vec{b}) = \Sigma i\left(\frac{\partial \vec{a}}{\partial x}\cdot\vec{b} + \vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \Sigma i\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right) + \Sigma i\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right). \quad \text{...}(i)$$

Now, $\vec{a} \times \left(i\times\frac{\partial \vec{b}}{\partial x}\right) = \left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i - (\vec{a}\cdot i)\frac{\partial \vec{b}}{dx}$

$$\Rightarrow \quad \left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \left(i\times\frac{\partial \vec{b}}{\partial x}\right) + (\vec{a}\cdot i)\frac{\partial \vec{b}}{\partial x}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \Sigma\vec{a} \times \left(i\times\frac{\partial \vec{b}}{\partial x}\right) + \Sigma(\vec{a}\cdot i)\frac{\partial \vec{b}}{\partial x}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \Sigma\left(i\times\frac{\partial \vec{b}}{\partial x}\right) + \Sigma\left(\vec{a}\cdot i\frac{\partial}{\partial x}\right)\vec{b}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \text{curl}\,\vec{b} + (\vec{a}\cdot\nabla)\,\vec{b}\,\cdot \quad \text{...}(ii)$$

Interchanging $\vec{a}$ and $\vec{a}$, we get

$$\Sigma\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right)i = \vec{b} \times \text{curl}\,\vec{a} + (\vec{b}\cdot\nabla)\,\vec{a} \quad \text{...}(iii)$$

From equations (*i*), (*ii*) and (*iii*), we get

$$\boxed{\text{grad }(\vec{a}\cdot\vec{b}) = \vec{a} \times \text{curl } \vec{b} + \vec{b} \times \text{curl } \vec{a} + (\vec{a}\cdot\nabla)\,\vec{b} + (\vec{b}\cdot\nabla)\,\vec{a}}.$$

Identity 3: $\text{div}\,(u\vec{a}) = u\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,u$

Proof:

$$\text{div}\,(u\vec{a}) = \nabla\cdot(u\vec{a})$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(u\vec{a})$$

$$= i\cdot\frac{\partial}{\partial x}(u\vec{a})+j\cdot\frac{\partial}{\partial y}(u\vec{a})+k\cdot\frac{\partial}{\partial z}(u\vec{a})$$

$$= i\cdot\left\{\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial \vec{a}}{\partial x}\right\}+j\cdot\left\{\frac{\partial u}{\partial y}\vec{a}+u\frac{\partial \vec{a}}{\partial y}\right\}+k\cdot\left\{\frac{\partial u}{\partial z}\vec{a}+u\frac{\partial \vec{a}}{\partial z}\right\}$$

$$= u\left\{i\cdot\frac{\partial \vec{a}}{\partial x}+j\cdot\frac{\partial \vec{a}}{\partial y}+k\cdot\frac{\partial \vec{a}}{\partial z}\right\}+\vec{a}\cdot\left\{i\frac{\partial u}{\partial x}+j\frac{\partial u}{\partial y}+k\frac{\partial u}{\partial z}\right\}$$

$$\boxed{\text{div}\,(u\vec{a}) = u\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,u}.$$

Identity 4: $\text{div}\,(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{curl}\,\vec{b} - \vec{a}\cdot\text{curl}\,\vec{b}$

Proof:

$$\text{div}\,(\vec{a}\times\vec{b}) = \nabla\cdot(\vec{a}\times\vec{b})$$

$$= \Sigma i\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$$

$$= \Sigma i\cdot\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b}+\vec{a}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \Sigma i\cdot\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b}\right)+\Sigma i\cdot\left(\vec{a}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \Sigma\left(i\times\frac{\partial \vec{a}}{\partial x}\right)\cdot\vec{b}-\Sigma\left(i\times\frac{\partial \vec{b}}{\partial x}\right)\cdot\vec{a}$$

$$= (\text{curl}\,\vec{a})\cdot\vec{b} - (\text{curl}\,b)\cdot\vec{a}$$

$$\boxed{\text{div}\,(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{curl}\,\vec{a} - \vec{a}\cdot\text{curl}\,\vec{b}}.$$

Identity 5: $\text{curl}\,(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}$

Proof:

$$\text{curl}\,(u\vec{a}) = \nabla\times(u\vec{a})$$

$$= \Sigma i\times\frac{\partial}{\partial x}(u\vec{a})$$

$$= \Sigma i\times\left(\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial \vec{a}}{\partial x}\right)$$

$$= \Sigma\left(i\frac{\partial u}{\partial x}\right)\times\vec{a}+u\Sigma\left(i\times\frac{\partial \vec{a}}{\partial x}\right)$$

$$= (\text{grad}\,u)\times\vec{a} + u\,\text{curl}\,\vec{a}$$

$$\boxed{\text{curl}\,(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}}.$$

Lecture 4

VECTOR INTEGRATION

Vector integral calculus extends the concepts of (ordinary) integral calculus to vector functions. It has applications in fluid flow design of under water transmission cables, heat flow in stars, study of satellites. Line integrals are useful in the calculation of work done by variable forces along paths in space and the rates at which fluids flow along curves (circulation) and across boundaries (flux).

LINE INTEGRAL

Let $\vec{F}(\vec{r})$ be a continuous vector point function. Then $\int_C \vec{F}\cdot d\vec{r}$, is known as the line integral of $\vec{F}(\vec{r})$ along the curve C.

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where F_1, F_2, F_3 are the components of $\vec{F}$ along the coordinate axes and are the functions of x, y, z each.

Now, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore$ $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (F_1 dx + F_2 dy + F_3 dz).$$

Again, let the parameteric equations of the curve C be

$$x = x(t)$$
$$y = y(t)$$
$$z = z(t)$$

then we can write $$\int_C \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx}{dt} + F_2(t)\frac{dy}{dt} + F_3(t)\frac{dz}{dt}\right]dt$$

were t_1 and t_2 are the suitable limits so as to cover the arc of the curve C.

Note: work done = $\int_C \vec{F}\cdot d\vec{r}$

Circulation: The line integral $\int_C \vec{F}\cdot d\vec{r}$ of a continuous vector point functional $\vec{F}$ along a closed curve C is called the circulation of $\vec{F}$ round the closed curve C.

Irrotational vector field: A single valued vector point Function $\vec{F}$ (Vector Field $\vec{F}$) is called irrotational in the region R, if its circulation round every closed curve C in that region is zero that is $\int_C \vec{F}\cdot d\vec{r} = 0$

Example . Evaluate $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$.

Sol. Let C denote the circle $x^2 + y^2 = 1$, *i.e.*, $x = \cos t$, $y = \sin t$. In order to integrate around C, t varies from 0 to 2π.

$$\therefore \quad \int_C (x\,dy - y\,dy) = \int_0^{2\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= \int_0^{2\pi} dt$$

$$= (t)_0^{2\pi}$$

$$= 2\pi.$$

Example Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = yz\,i + zx\,j + xy\,k$ and C is the portion of the curve $\vec{r} = (a \cos t)\,i + (b \sin t)\,j + (ct)\,k$ from $t = 0$ to $\frac{\pi}{2}$.

Sol. We have $\quad \vec{r} = (a \cos t)\,i + (b \sin t)\,j + (ct)\,k.$

Hence, the parametric equations of the given curve are $x = a \cos t \quad y = b \sin t \quad z = ct$

Also, $$\frac{d\vec{r}}{dt} = (-a \sin t)\,i + (b \cos t)\,j + ck$$

Now, $$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

$$= \int_C (yzi + zxj + xyk) \cdot (-a\sin t\,i + b\cos t\,j + ck) dt$$

$$= \int_C (bct \sin t\,i + act\cos t\,j + ab\sin t\cos t\,k) \cdot (-a\sin t\,i + b\cos t\,j + ck) dt$$

$$= \int_C \left(-abc\,t\sin^2 t + abc\,t\cos^2 t + abc\sin t\cos t \right) dt$$

$$= abc\int_C \left[t\left(\cos^2 t - \sin^2 t\right) + \sin t \cos t \right] dt = abc\int_C \left(t\cos 2t + \frac{\sin 2t}{2} \right) dt$$

$$= abc\int_0^{\frac{\pi}{2}} \left(t\cos 2t + \frac{\sin 2t}{2} \right) dt = abc\left[t\frac{\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{\cos 2t}{4} \right]_0^{\frac{\pi}{2}}$$

Lecture 5

SURFACE INTEGRAL

Any integral which is to be evaluated over a surface is called a surface integral.

Let $\vec{F}(\vec{r})$ be a continuous vector point function. Let $\vec{r} = \vec{F}(u, v)$ be a smooth surface such that $\vec{F}(u, v)$ possesses continuous first order partial derivatives. Then the normal surface integral of $\vec{F}(\vec{r})$ over S is denoted by

$$\int_S \vec{F}(\vec{r}) \cdot d\vec{a} = \int_S \vec{F}(\vec{r}) \cdot \hat{n} dS$$

where $d\vec{a}$ is the vector area of an element dS and $\hat{n}$ is a unit vector normal to the surface dS.

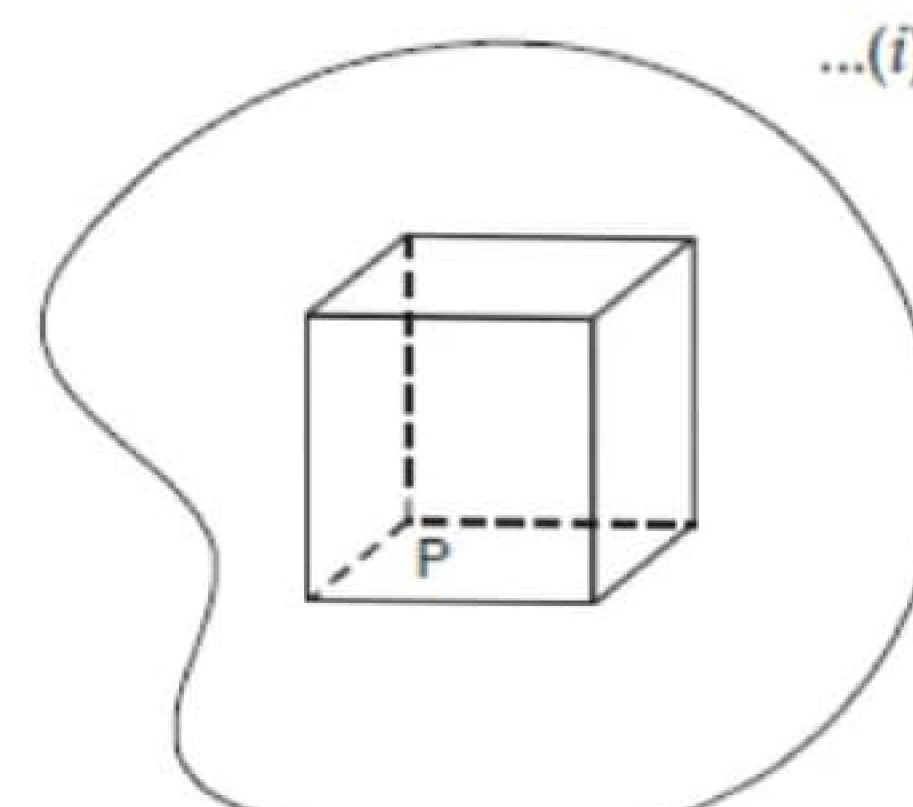
Let F_1, F_2, F_3 which are the functions of x, y, z be the components of F along the coordinate axes, then

$$\begin{aligned}
\text{Surface Integral} &= \int_S \vec{F} \cdot \hat{n} dS \\
&= \int_S \vec{F} \cdot d\vec{a} \\
&= \int\int_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k}) \\
&= \int\int_S (F_1 dy\, dz + F_2 dz\, dx + F_3 dx\, dy).
\end{aligned}$$

VOLUME INTEGRAL

Let $\vec{F}(\vec{r})$ is a continuous vector point function. Let volume V be enclosed by a surface S given by

$$\vec{r} = \vec{f}(u,v) \qquad \text{...(i)}$$



sub-dividing the region V into n elements say of cubes having volumes

$$\Delta V_1, \Delta V_2, \text{....} \Delta V_n$$

Hence

$$\Delta V_k = \Delta x_k \, \Delta y_k \, \Delta z_k$$
$$k = 1, 2, 3, \dots n$$

where (x_k, y_k, z_k) is a point say P on the cube. Considering the sum

$$\sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k)\Delta V_k$$

taken over all possible cubes in the region. The limits of sum when $n \to \infty$ in such a manner that the dimensions ΔV_k tends to zero, if it exists is denoted by the symbol

$$\int_V \vec{F}(\vec{r})dV \cdot or \int_V \vec{F}dV \;\; or \int\int\int_V \vec{F}\, dx\, dy\, dz$$

is called volume integral or space integral.

Example Evaluate $\iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) \cdot d\vec{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

Sol.

$$\iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) \cdot d\vec{S} = \iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) \cdot \left(dy\ dz\ \hat{i} + \cdot dz\ dx\ \hat{j} + dx\ dy\ \hat{k}\right)$$

$$= \iint_S (yz\ dy\ dz + zx\ dz\ dx + xy\ dx\ dy)$$

$$= \int_0^a \int_0^{\sqrt{a^2 - z^2}} yz\ dy\ dz + \int_0^a \int_0^{\sqrt{a^2 - x^2}} zx\ dz\ dx + \int_0^a \int_0^{\sqrt{a^2 - y^2}} xy\ dx\ dy$$

$$= \int_0^a z\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-z^2}} dz + \int_0^a x\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx + \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a z(a^2 - z^2)dz + \frac{1}{2}\int_0^a x(a^2 - x^2)dx + \frac{1}{2}\int_0^a y(a^2 - y^2)dy$$

$$= \frac{1}{2}\left(\frac{a^2z^2}{2} - \frac{z^4}{4}\right)_0^a + \frac{1}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a + \frac{1}{2}\left(\frac{a^2y^2}{2} - \frac{y^4}{4}\right)_0^a$$

$$= \frac{1}{2}\frac{a^4}{4} + \frac{1}{2}\frac{a^4}{4} + \frac{1}{2}\frac{a^4}{4} = \frac{3a^4}{8}.$$

Example : Evaluate $\iint \vec{F} \cdot \hat{n}\, ds$, where $\vec{F} = zi + xj - 3y^2 zk$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol. Since surface $S : x^2 + y^2 = 16$

Let $f \equiv x^2 + y^2 - 16$

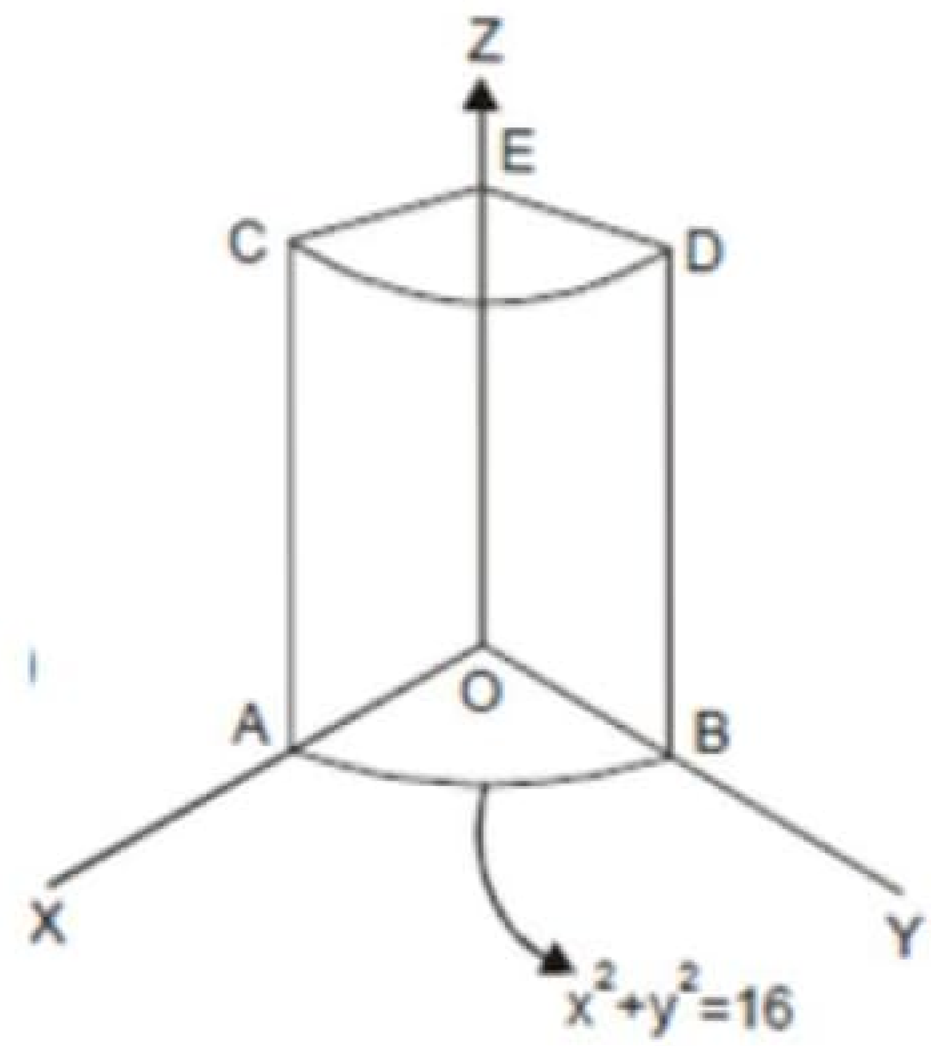
$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 - 16)$$

$$= 2xi + 2yi$$

unit normal $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2xi + 2yj}{\sqrt{4x^2 + 4y^2}}$

$$\hat{n} = \frac{2(xi + yj)}{2\sqrt{x^2 + y^2}} = \frac{xi + yj}{\sqrt{16}} = \frac{xi + yi}{4}$$

Now $\vec{F} \cdot \hat{n} = (zi + xj - 3y^2 zk) \cdot \left(\dfrac{xi + yj}{4}\right) = \dfrac{1}{4}(zx + xy)$



Here the surface S is perpendicular to xy-plane so we will take the projection of S on zx-plane. Let R be that projection.

$$\therefore \qquad ds = \frac{dx\,dz}{|\hat{n}\cdot j|} = \frac{dx\,dz}{\frac{y}{4}} = \frac{4dx\,dz}{y}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \left(zi + xj - 3y^2zk\right)\cdot\frac{(xi+yj)}{4}\cdot\frac{4}{y}dx\,dz$$

$$= \iint_R \left(\frac{zx+xy}{y}\right)dx\,dz$$

Since z varies from 0 to 5 and $y = \sqrt{16-x^2}$ on S. x is also varies from 0 to 4.

$$\therefore \qquad \iint_R \left(\frac{zx+xy}{y}\right)dx\,dz = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}} + x\right)dx\,dz$$

$$= \int_0^5 \left[-z\sqrt{16-x^2} + \frac{x^2}{2}\right]_0^4 dz = \int_0^5 (4z+8)\,dz$$

$$= (2z^2 + 8z)_0^5 = 50 + 40 = 90.$$

Lecture 6

GREEN'S* THEOREM

If C be a regular closed curve in the xy-plane bounding a region S and $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions inside and on C then

$$\iint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

Example : Verify Green's theorem in plane for $\oint_C (x^2 - 2xy)dx + (x^2 y + 3)dy$, where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

Sol. By Green's theorem

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

i.e., Line Integral (*LI*) = Double Integral (*DI*)

Here, $P = x^2 - 2xy,\ Q = x^2 y + 3$

$$\frac{\partial P}{\partial y} = -2x, \frac{\partial Q}{\partial x} = 2xy$$

So the R.H.S. of the Green's theorem is the double integral given by

$$DI = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_S [(2xy - (-2x))]\, dx\, dy$$

The region S is covered with y varying from $-2\sqrt{2}\sqrt{x}$ of the lower branch of the parabola to its upper branch $2\sqrt{2}\sqrt{x}$ while x varies from 0 to 2. Thus

$$DI = \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\, dy\, dx$$

$$= \int_0^2 xy^2 + 2xy \Big|_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= 8\sqrt{2} \int_0^2 x^{\frac{3}{2}} dx = \frac{128}{5}$$

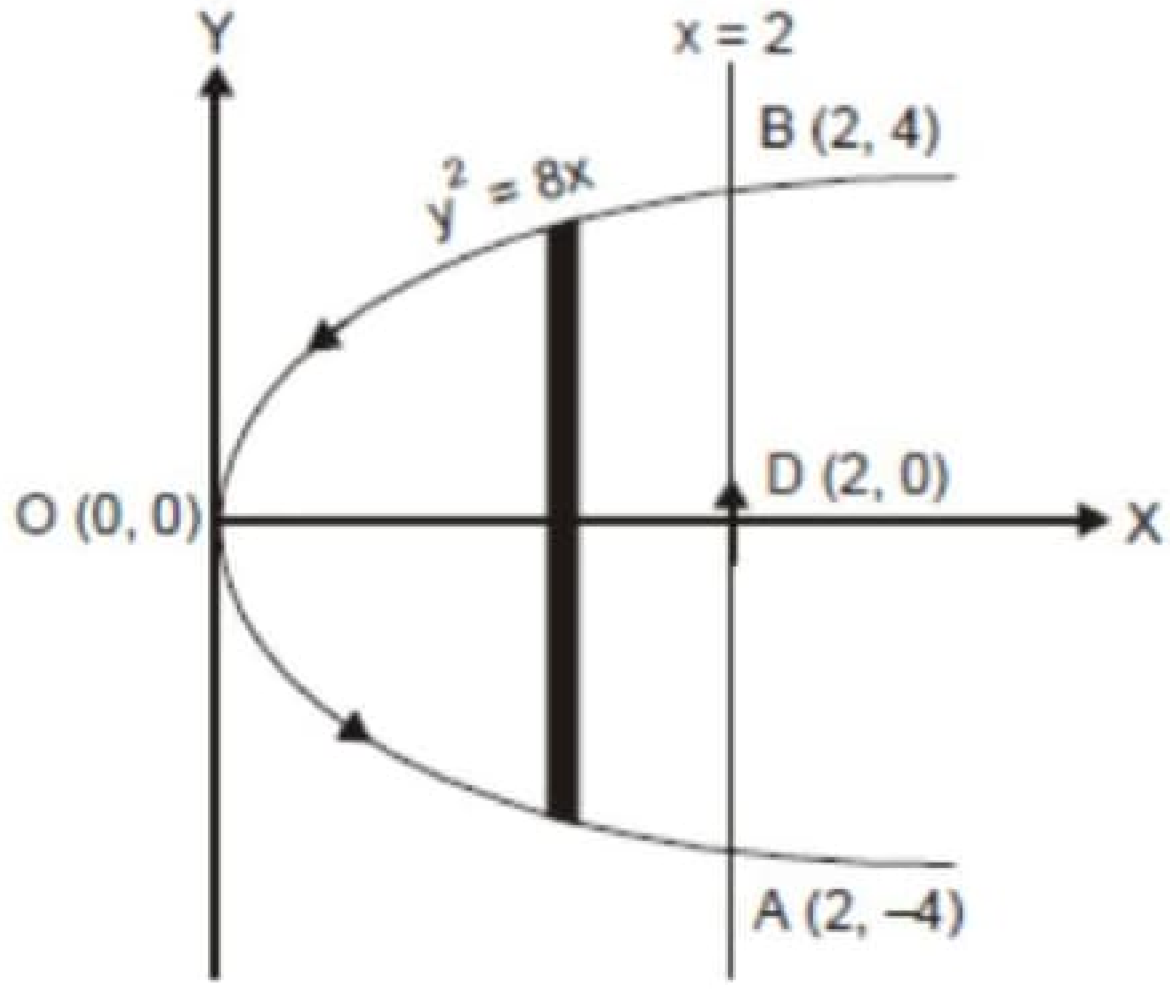
The L.H.S. of the Green's theorem result is the line integral

$$LI = \oint_C \left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy \cdot$$

Here C consists of the curves OA, ADB. BO. so

$$LI = \oint_C = \int_{OA+ADB+BO}$$

$$= \int_{OA} + \int_{ADB} + \int_{BO} = LI_1 + LI_2 + LI_3$$



Along OA: $y = -2\sqrt{2}\sqrt{x}$, so $dy = -\sqrt{\frac{2}{x}}dx$

$$LI_1 = \int_{OA}\left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy$$

$$= \int_0^2 \left[x^2 - 2x\left(-2\sqrt{2}\sqrt{x}\right)\right]dx$$

$$+\left[x^2\left(-2\sqrt{2}\sqrt{x}\right) + 3\right]\left(-\sqrt{\frac{2}{x}}\right)dx$$

$$= \int_0^2 \left(5x^2 + 4\sqrt{2}\cdot x^{3/2} - 3\sqrt{2}x^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{5x^3}{3} + 4\sqrt{2}\,\frac{2}{5}x^{\frac{5}{2}} - 3\sqrt{2}\cdot 2\sqrt{x}\right]_0^2$$

$$= \frac{40}{3} + \frac{64}{5} - 12$$

Along ADB **:** $x = 2,\ dx = 0$

$$LI_2 = \int_{ADB} (x^2 - 2xy)dx + (x^2y + 3)dy$$

$$= \int_{-4}^{4} (4y + 3)dy = 24$$

Along BO **:** $y = 2\sqrt{2}\sqrt{x}$, with $x : 2$ to 0.

$$dy = \sqrt{\frac{2}{x}}dx$$

$$LI_3 = \int_{BO} (x^2 - 2xy)dx + (x^2y + 3)dy$$

$$= \int_2^0 \left(5x^2 - 4\sqrt{2}x^{\frac{3}{2}} + 3\sqrt{2}x^{-\frac{1}{2}}\right)dx$$

$$= -\frac{40}{3} + \frac{64}{5} - 12$$

$$LI = LI_1 + LI_2 + LI_3 = \left(\frac{40}{3} + \frac{64}{5} - 12\right) + (24) + \left(-\frac{40}{3} + \frac{64}{5} - 12\right) = \frac{128}{5}$$

$\Rightarrow$ Hence the Green's theorem is verified.

Lecture 7

STOKE'S THEOREM

If $\vec{F}$ is any continuously differentiable vector function and S is a surface enclosed by a curve C, then

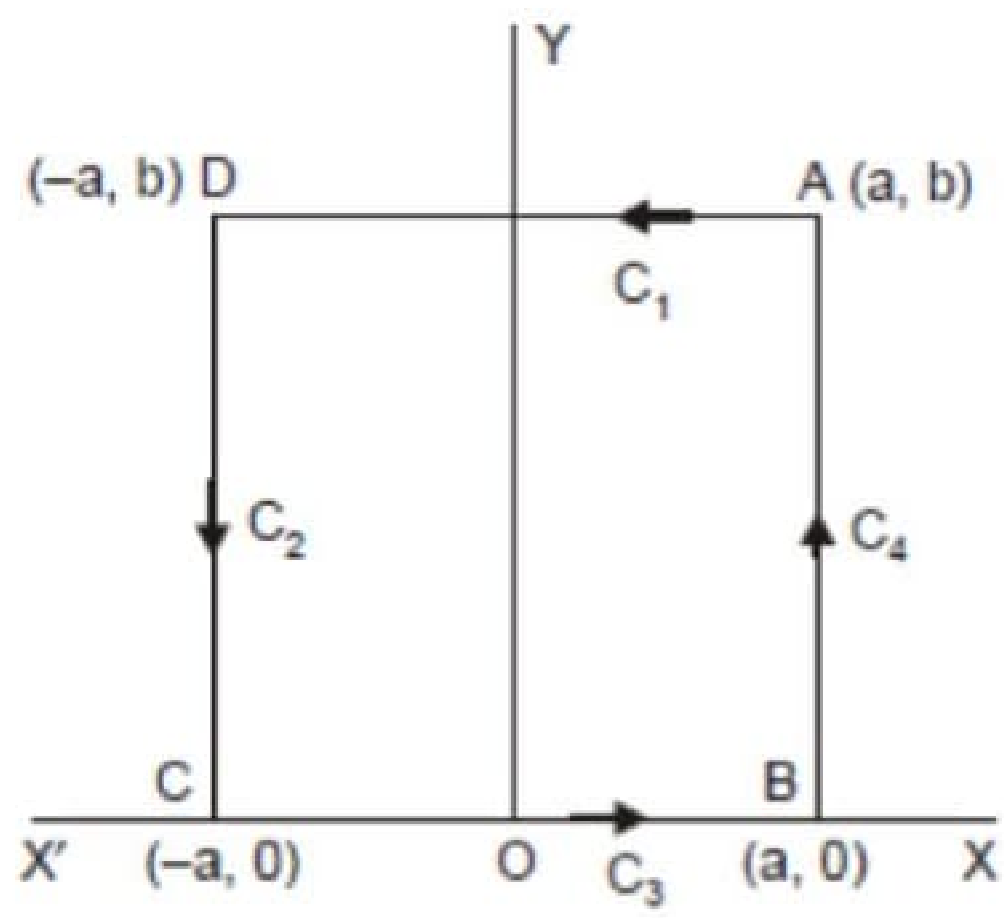
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS.$$

where $\hat{n}$ is the unit normal vector at any point of S.

Example Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\,i - 2xy\,j$ taken round the rectangle bounded by $x = \pm a,\ y = 0,\ y = b.$

Sol. We have $\vec{F}\cdot dr = \{(x^2 + y^2)\,i - 2xy\,j\}\cdot\{dx\,i + dy\,j\}$

$$= (x^2 + y^2)\,dx - 2xy\,dy$$



$$\therefore\quad \int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r}$$

$$= I_1 + I_2 + I_3 + I_4$$

$$\therefore\quad I_1 = \int_{C_1}\{(x^2+y^2)dx - 2xy\,dy\}$$

$$= \int_a^{-a}\{(x^2+b^2)\,dx - 0\} \qquad \begin{bmatrix}\because y = b\\ \therefore dy = 0\end{bmatrix}$$

$$= \left(\frac{x^3}{3} + b^2x\right)_a^{-a}$$

$$= -\left(\frac{2}{3}a^3 + 2b^2a\right)$$

$$I_2 = \int_{C_2}\{(x^2+y^2)dx - 2xy\,dy\}$$

$$= \int_b^0 \{(-a)^2 + y^2\}\,0 - 2(-a)\,y\,dy \qquad \begin{bmatrix}\because x = -a\\ \therefore dx = 0\end{bmatrix}$$

$$= 2a\int_b^0 y\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_b^0 = -\,ab^2$$

$$I_3 = \int_{C_3}(x^2+y^2)dx - 2xy\,dy = \int_{C_3} x^2dx \qquad \begin{bmatrix}\because y = 0\\ \therefore dy = 0\end{bmatrix}$$

$$= \int_{-a}^{+a} x^2dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{2a^3}{3}$$

$$I_4 = \int_{C_4} -2ay\,dy$$

$$= -2a \int_0^b y\,dy = -2a\left(\frac{y^2}{2}\right)_0^b$$

$$= -ab^2$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3 + I_4$$

$$= -\left(\frac{2a^3}{3} + 2b^2a\right) - ab^2 + \frac{2}{3}a^3 - ab^2$$

$$= -4ab^2 \qquad \text{...}(i)$$

Again,

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= -4yk$$

$$\hat{n} = k$$

$$\therefore \quad \hat{n}\cdot \text{curl } \vec{F} = k\cdot(-4yk) = -4y$$

$$\therefore \quad \iint_S \hat{n}\cdot \text{curl } \vec{F}\, dS = \int_{-a}^{a}\int_0^b -4y\,dx\,dy$$

$$= \int_{-a}^{+a} -4\left(\frac{y^2}{2}\right)_0^b dx$$

$$= -2b^2(x)_{-a}^{a}$$

$$= -4ab^2. \qquad \text{...}(ii)$$

From eqns. (*i*) and (*ii*), we verify Stoke's theorem.

Lecture 8

GAUSS'S DIVERGENCE THEOREM

If $\vec{F}$ is a continuously differentiable vector point function in a region V and S is the closed surface enclosing the region V, then

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV \qquad \text{...(i)}$$

where $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Example Evaluate $\iint_S \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}\,dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane.

Sol. Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}\left(y^2z^2\right) + \frac{\partial}{\partial y}\left(z^2x^2\right) + \frac{\partial}{\partial z}\left(z^2y^2\right)\right]dV$$

$$= \iiint_V 2zy^2\,dV = 2\iiint_V zy^2\,dV$$

Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore \quad 2\iiint_V zy^2\,dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)\left(r^2\sin^2\theta\sin^2\phi\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\,d\phi$$

$$= \frac{1}{12}\int_0^{2\pi}\sin^2\phi\cdot d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\,d\phi = \frac{\pi}{12}.$$